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The leading non-perturbative coefficient in the weak-coupling expansion of hot QCD pressure

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Abstract: Using Numerical Stochastic Perturbation Theory within three-dimensional pure SU(3) gauge theory, we estimate the last unknown renormalization constant that is needed for converting the vacuum energy density of this model from lattice regularization to the $\overline{\rm MS}$ scheme. Making use of a previous non-perturbative lattice measurement of the plaquette expectation value in three dimensions, this allows us to approximate the first non-perturbative coefficient that appears in the weak-coupling expansion of hot QCD pressure.

Keywords: Thermal Field Theory, Lattice Gauge Field Theories, Nonperturbative Effects, NLO Computations.

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1. Introduction

It is well known that, despite asymptotic freedom, QCD can display non-perturbative phenomena even in situations where the system is characterised by a large momentum or mass scale Q, $Q \gg 1$ GeV. For instance, in deep inelastic scattering, power-suppressed contributions may arise, of the form $\sim E_{\rm QCD}^n/Q^n$, where the numerator represents a non-perturbative contribution related to a certain quark or gluon condensate of dimension n, arising from the Operator Product Expansion, and $E_{\rm QCD}$ denotes the typical QCD energy scale, of a few hundred MeV.

A conceptually similar situation arises when Q is replaced by a high temperature T, and the observable is replaced by minus the grand canonical free energy density, or the pressure, $p(T)$. The Operator Product Expansion gets then replaced by a construction of a low-energy effective field theory, which in the case of high temperatures amounts to dimensional reduction [1]. The non-perturbative scale $E_{\rm QCD}$ gets replaced by that of the effective theory, $\sim \alpha_s T$ [2, 3], where $\alpha_s = g^2/4\pi$ is the strong coupling constant. In the effective theory, the leading non-perturbative condensate has the dimension $n = 3$. Therefore, the formal weak-coupling expansion of $p(T)$ contains non-perturbative coefficients, starting at $\mathcal{O}(g^6)$ [2].

While $\mathcal{O}(g^6)$ may appear to be an academically high order, recent advances in perturbative QCD have made its determination an issue of practical importance [4, 5]. Indeed, perturbative corrections to the non-interacting Stefan-Boltzmann form of $p(T)$ have been determined at relative orders $\mathcal{O}(g^2)$ [6], $\mathcal{O}(g^3)$ [7], $\mathcal{O}(g^4\ln(1/g))$ [8], $\mathcal{O}(g^4)$ [9], $\mathcal{O}(g^5)$ [10, 11], and $\mathcal{O}(g^6\ln(1/g))$ [12], as a function of the number of colours, $N_{\rm c}$, the number of massless

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quark flavours, $N_{\rm f}$, and, most recently, the chemical potentials, μ_i, that can be assigned to the various quark flavours [13] (as long as they are small enough compared with the temperature, $|\mu_i| \lesssim T/g$ [14]). Reaching the next unknown order $\mathcal{O}(g^6)$ depends, therefore, on the inclusion of the non-perturbative term. Moreover, studies with high-temperature observables other than the pressure, in which analogous non-perturbative coefficients arise but at lower orders, have shown that their inclusion is in general numerically important, in order to reliably determine the properties of hot QCD at physically relevant temperatures [15, 16].

By now, the first step has already been taken in order to determine the non-perturbative $\mathcal{O}(g^6)$ term: the gluon condensate of dimension $n = 3$ was measured with lattice Monte Carlo techniques within the dimensionally reduced effective field theory in Ref. [17]. Given that the other parts of the pressure computation have been formulated in the continuum $\overline{\rm MS}$ scheme, however, this result needs still to be converted to the same regularization [18, 19]. The purpose of the present paper is to finalise this task with a certain numerical precision. Afterwards, only purely perturbative contributions in the $\overline{\rm MS}$ scheme remain to be determined, in order to complete the expression of $p(T)$ up to $\mathcal{O}(g^6)$ (in the notation of Ref. [12], $\beta_{\rm E1}$ remains unknown, while $\beta_{\rm E2}, \beta_{\rm E3}$ have recently become available [20, 16]).

We note that apart from formal interests, the computations outlined above may also have phenomenological relevance, in the contexts of cosmology and of heavy ion collision experiments. We do not elaborate on these issues in the present paper, however, since the current status on these fronts has very recently been summarised elsewhere [21].

The plan of this paper is the following. In section 2 we set up the notation and describe the overall strategy of the computation. Section 3 contains a discussion of our computational tool, Numerical Stochastic Perturbation Theory in a covariant gauge. Section 4 contains the data analysis and our results, while section 5 draws some conclusions.

2. Basic definitions and overall strategy

Let us start by considering three-dimensional (3d) pure SU($N_{\rm c}$) gauge theory in dimensional regularization. The Euclidean continuum action can be written as

$$S_{\rm E} = \int {\rm d}^d x \, \mathcal{L}_{\rm E} \; , \qquad \mathcal{L}_{\rm E} = \frac{1}{2 g_3^2} \, {\rm Tr} \, [F_{kl} F_{kl}] \; , \tag{2.1}$$

where $d = 3 - 2\epsilon$, g_3^2 is the (dimensionful) gauge coupling, $k, l = 1, \ldots, d$, $F_{kl} = i[D_k, D_l]$, $D_k = \partial_k + i A_k$, $A_k = A_k^B T^B$, T^B are the Hermitean generators of SU($N_{\rm c}$), normalised as ${\rm Tr} \, [T^A T^B] = \delta^{AB}/2$, and repeated indices are assumed to be summed over. Leaving out for brevity gauge fixing and Faddeev-Popov terms, the "vacuum energy density" is defined as

$$f_{\overline{\rm MS}} \equiv - \lim_{V \to \infty} \frac{1}{V} \ln \biggl[\int \mathcal{D} A_k \, \exp\Bigl(- S_{\rm E} \Bigr) \biggr]_{\overline{\rm MS}} \; , \tag{2.2}$$

where V is the d-dimensional volume, $\mathcal{D} A_k$ a (gauge-invariant) functional integration measure, and we have assumed the use of the $\overline{\rm MS}$ dimensional regularization scheme to remove any $1/\epsilon$ poles from the expression.

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The physical significance of $f_{\overline{\rm MS}}$ for high-temperature QCD enters through the framework of dimensional reduction [1, 22]. Indeed, after the replacement $g_3^2 \to g^2 T[1 + \mathcal{O}(g^2)]$, $f_{\overline{\rm MS}}$ appears in the QCD pressure as an additive contribution, $\delta p(T) = -T f_{\overline{\rm MS}}$ [2, 11]. The dependence on regularization (in particular, on the $\overline{\rm MS}$ scheme scale parameter $\bar{\mu}$) disappears, once all other contributions of the same order [12] have been added.

Now, dimensional reasons and a perturbative computation of ultraviolet divergences [23] show that the structure of $f_{\overline{\rm MS}}$ is (after letting $\epsilon \to 0$)

$$f_{\overline{\rm MS}} = -g_3^6 \frac{d_A N_{\rm c}^3}{(4\pi)^4} \left[\left(\frac{43}{12} - \frac{157}{768}\pi^2 \right) \ln \frac{\bar{\mu}}{2N_{\rm c} g_3^2} + B_{\rm G} \right] , \tag{2.3}$$

where $d_A \equiv N_{\rm c}^2 - 1$. The non-perturbative constant $B_{\rm G}$, which actually is a function of $N_{\rm c}$, is what we would like to estimate in the following. For future reference, we note that a logarithmic derivative of $f_{\overline{\rm MS}}$ with respect to g_3^2 immediately produces the gluon condensate:

$$\frac{1}{2g_3^2} \Big\langle {\rm Tr}\, [F_{kl} F_{kl}] \Big\rangle_{\overline{\rm MS}} = 3 g_3^6 \frac{d_A N_{\rm c}^3}{(4\pi)^4} \left[\left(\frac{43}{12} - \frac{157}{768}\pi^2 \right) \left(\ln \frac{\bar{\mu}}{2N_{\rm c} g_3^2} - \frac{1}{3} \right) + B_{\rm G} \right] . \tag{2.4}$$

We now go to lattice regularization. In standard Wilson discretization, the lattice action, $S_{\rm L}$, corresponding to eq. (2.1), reads

$$S_{\rm L} = \beta \sum_{\mathbf{x}} \sum_{k<l} \Big[1 - \Pi_{kl}(\mathbf{x}) \Big] , \tag{2.5}$$

where $\Pi_{kl}(\mathbf{x}) \equiv {\rm Re}[{\rm Tr}\, U_{kl}(\mathbf{x})]/N_{\rm c}$, $U_{kl}(\mathbf{x}) \equiv U_k(\mathbf{x}) U_l(\mathbf{x}+k) U_k^\dagger(\mathbf{x}+l) U_l^\dagger(\mathbf{x})$ is the plaquette, $U_k(\mathbf{x})$ is a link matrix, $\mathbf{x} + k \equiv \mathbf{x} + a\hat{\epsilon}_k$, where a is the lattice spacing and $\hat{\epsilon}_k$ is a unit vector, and

$$\beta \equiv \frac{2N_{\rm c}}{g_3^2 a} . \tag{2.6}$$

Note that the gauge coupling does not get renormalised in 3d, and the parameters g_3^2 appearing in eqs. (2.1), (2.6) can hence be assumed finite and equivalent. The observable we consider is still the vacuum energy density, eq. (2.2), which in lattice regularization reads

$$f_{\rm L} \equiv - \lim_{V\to\infty} \frac{1}{V} \ln \left[\int \mathcal{D}U_k \, \exp\Big(-S_{\rm L} \Big) \right] , \tag{2.7}$$

where $\mathcal{D}U_k$ denotes integration over link matrices with the gauge-invariant Haar measure.

Being in principle physical quantities, the values of $f_{\overline{\rm MS}}$ and $f_{\rm L}$ must agree, provided that suitable vacuum counterterms are added to the theory. Due to super-renormalizability, there can be such counterterms up to 4-loop level only [24], and correspondingly

$$\begin{aligned} \Delta f &\equiv f_{\rm L} - f_{\overline{\rm MS}} \\ &= C_1 \frac{1}{a^3} \left(\ln \frac{1}{a g_3^2} + C_1' \right) + C_2 \frac{g_3^2}{a^2} + C_3 \frac{g_3^4}{a} + C_4 g_3^6 \left(\ln \frac{1}{a\bar{\mu}} + C_4' \right) + \mathcal{O}(g_3^8 a) , \end{aligned} \tag{2.8}$$

where the C_i are dimensionless functions of $N_{\rm c}$. The values of C_1, C_2, C_3, C_4 are known, as we will recall presently; C_1' is related to the precise normalisation of the Haar integration measure and has no physical significance; and C_4' will be estimated below.

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Correspondingly, the gluon condensates, i.e. the logarithmic derivatives of $f_{\overline{\rm MS}}$, $f_{\rm L}$ with respect to g_3^2, can also be related by a perturbative 4-loop computation. Noting that three-dimensional rotational and translational symmetries allow us to write

$$-g_3^2\frac{\partial}{\partial g_3^2}f_{\rm L} = \frac{3\beta}{a^3}\langle 1-\Pi_{12}\rangle_{\rm L}\;, \tag{2.9}$$

and employing eqs. (2.4), (2.8), leads to [17]

$$8\frac{d_A N_{\rm c}^6}{(4\pi)^4}B_{\rm G} = \lim_{\beta\to\infty}\beta^4\biggl\{\langle 1-\Pi_{12}\rangle_{\rm L} - \biggl[\frac{c_1}{\beta}+\frac{c_2}{\beta^2}+\frac{c_3}{\beta^3}+\frac{c_4}{\beta^4}\Bigl(\ln\beta+c_4'\Bigr)\biggr]\biggr\}\;. \tag{2.10}$$

The values of the constants $c_1,\ldots,c_4'$ are trivially related to those of $C_1,\ldots,C_4'$ in eq. (2.8): $c_1 = C_1/3$, $c_2 = -2N_{\rm c}C_2/3$, $c_3 = -8N_{\rm c}^2C_3/3$, $c_4 = -8N_{\rm c}^3C_4$ and, in particular,

$$c_4' = C_4' - \frac{1}{3} - 2\ln(2N_{\rm c})\;. \tag{2.11}$$

For $N_{\rm c}=3$, the constants read [17, 25–27]

$$c_1 = \frac{d_A}{3} \approx 2.66666667\;, \tag{2.12}$$

$$c_2 = 1.951315(2)\;, \tag{2.13}$$

$$c_3 = 6.8612(2)\;, \tag{2.14}$$

$$c_4 = 8\frac{d_A N_{\rm c}^6}{(4\pi)^4}\left(\frac{43}{12}-\frac{157}{768}\pi^2\right) \approx 2.92942132\;. \tag{2.15}$$

Moreover, lattice measurements [17] have shown that, for $N_{\rm c}=3$,

$$B_{\rm G} + \left(\frac{43}{12}-\frac{157}{768}\pi^2\right)c_4' = 10.7\pm 0.4\;. \tag{2.16}$$

In order to extract $B_{\rm G}$, as is our goal, we need to determine the unknown constant c_4' in eq. (2.16). This can be achieved by repeating the same setup as above, but by regulating infrared (IR) divergences through a mass regulator, m, instead of confinement. Indeed, the difference in eq. (2.8) is IR insensitive, and does not change. Thus we can extract C_4' this way and, from eq. (2.11), c_4'. We denote quantities computed with a mass regulator with a tilde.

With a mass regulator, the continuum computation produces [23]

$$\tilde{f}_{\overline{\rm MS}} = \cdots - g_3^6\frac{d_A N_{\rm c}^3}{(4\pi)^4}\left[\left(\frac{43}{12}-\frac{157}{768}\pi^2\right)\ln\frac{\bar{\mu}}{2m}+\tilde{B}_{\overline{\rm MS}}(\alpha)\right]+\mathcal{O}\Bigl(\frac{g_3^8}{m}\Bigr)\;, \tag{2.17}$$

where the lower order terms omitted vanish for $m\to 0$, and $\tilde{B}_{\overline{\rm MS}}$ depends on the gauge parameter α, since the introduction of a mass breaks gauge invariance. If, on the other hand, we carry out the same computation in lattice regularization, we expect

$$\tilde{f}_{\rm L} = \cdots - g_3^6\frac{d_A N_{\rm c}^3}{(4\pi)^4}\left[\left(\frac{43}{12}-\frac{157}{768}\pi^2\right)\ln\frac{1}{am}+\tilde{B}_{\rm L}(\alpha)+\mathcal{O}(ma)\right]+\mathcal{O}\Bigl(g_3^8 a,\frac{g_3^8}{m}\Bigr)\;, \tag{2.18}$$

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where the lower order terms omitted go over to the ones in eq. (2.8) for $m \to 0$. Given these forms and the IR insensitivity of the difference in eq. (2.8), we get

$$\begin{aligned} \Delta f &= \lim_{m\to 0}\Big[\tilde{f}_{\rm L} - \tilde{f}_{\overline{\rm MS}}\Big] \\ &= \cdots - g_3^6 \frac{d_A N_{\rm c}^3}{(4\pi)^4}\left[\left(\frac{43}{12} - \frac{157}{768}\pi^2\right)\ln\frac{2}{a\bar{\mu}} + \tilde{B}_{\rm L}(\alpha) - \tilde{B}_{\overline{\rm MS}}(\alpha)\right] + \mathcal{O}(g_3^8 a)\;, \end{aligned} \tag{2.19}$$

where the lower order terms omitted agree with eq. (2.8). Comparing the $\mathcal{O}(g_3^6)$ terms with eq. (2.8), we can read off C_4'. Subsequently, eq. (2.11) gives c_4'. Inserting finally the result from eq. (2.16), we arrive at the master relation

$$B_{\rm G} = 10.7 \pm 0.4 - \tilde{B}_{\rm L}(\alpha) + \tilde{B}_{\overline{\rm MS}}(\alpha) + \left(\frac{43}{12} - \frac{157}{768}\pi^2\right)\left(\frac{1}{3} + \ln 2 + 2\ln N_{\rm c}\right)\,. \tag{2.20}$$

It remains to determine $\tilde{B}_{\overline{\rm MS}}(\alpha)$ and $\tilde{B}_{\rm L}(\alpha)$. Since the result of eq. (2.20) is gauge independent, we choose the covariant Feynman gauge ($\alpha = 1$) in the following. A 4-loop continuum computation, described in Ref. [23], leads to a small set of fully massive master integrals that are known with high precision [28], and produces (see also Ref. [29])

$$\tilde{B}_{\overline{\rm MS}}(1) = -2.165625919498009 19016\ldots\;. \tag{2.21}$$

On the other hand, taking a logarithmic derivative with respect to g_3^2 from eq. (2.18), we obtain, in complete analogy with eq. (2.10),

$$8\frac{d_A N_{\rm c}^6}{(4\pi)^4}\tilde{B}_{\rm L}(\alpha) = \lim_{m\to 0}\beta^4\left\{\langle 1 - \tilde{\Pi}_{12}\rangle_{\rm L}\Big|_{\text{up to 4-loop}} - \left[\frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \frac{c_3}{\beta^3} + \frac{c_4}{\beta^4}\ln\frac{1}{am}\right]\right\}\,. \tag{2.22}$$

Our task in the following is to compute the right-hand side of this equation for $\alpha = 1$ and, afterwards, to insert the result into eq. (2.20), in order to estimate $B_{\rm G}$.

3. Numerical stochastic perturbation theory

To carry out the limit in eq. (2.22) requires a 4-loop computation in lattice perturbation theory. As the master integrals that appear in higher loop computations in lattice regularization need to be evaluated numerically in any case, we choose to carry out the whole computation numerically. This can be achieved through the use of Numerical Stochastic Perturbation Theory (NSPT) [30], pioneered in recent years by the Parma group; a full account of the method can be found in Ref. [31].

In its "purest" form, NSPT can be applied without either gauge fixing or masses as IR regulators. Since we compare with a dimensionally regularized gauge fixed continuum computation with a mass as an IR regulator, however, we need to introduce the same tools in NSPT. The first two subsections describe our general implementation, and the third collects some technical details of the computation.

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3.1 NSPT in a covariant gauge

NSPT relies on Stochastic Quantization [32] (for an extensive review, see Ref. [33]). In this approach quantum fields are given an extra coordinate, τ, which is to be regarded as a stochastic time in which an evolution takes place according to the Langevin equation. This is in close analogy with the "time" evolution of the Markov chain that is used in standard Monte Carlo simulations; indeed, Stochastic Quantization can also be used for Monte Carlo simulations [34] (for a concise review, see Ref. [35]).

For lattice gauge theories, the Langevin equation reads

$$\partial_\tau U_{k,\eta}(\mathbf{x},\tau) = -i\Big\{\nabla_{k,\mathbf{x}} S[U_{k,\eta}] + \eta_k(\mathbf{x},\tau)\Big\}\, U_{k,\eta}(\mathbf{x},\tau)\ , \tag{3.1}$$

where we assume the use of lattice units ($a = 1$) in the spatial directions. The derivative $\nabla_{k,\mathbf{x}}$ is defined [36] as $\nabla_{k,\mathbf{x}} \equiv T^A \nabla^A_{k,\mathbf{x}}$, with $\nabla^A_{k,\mathbf{x}} S[U_k(\mathbf{x})] \equiv \lim_{\epsilon\to 0}\{S[\exp(i\epsilon T^A)U_k(\mathbf{x})] - S[U_k(\mathbf{x})]\}/\epsilon$, where T^A are the generators in the fundamental representation, normalised as before. Moreover, η_k is a gaussian noise in the adjoint representation, $\eta_k \equiv \eta_k^A T^A$. In eq. (3.1) we adhere to a precise notation in which the dependence of the solution on the stochastic noise is explicitely shown. Since this notation is a bit pedantic, we will drop it in the following. It is worth stressing that the evolution dictated by eq. (3.1) preserves unitarity, and that $\nabla_{k,\mathbf{x}}$ is consistent with partial integration over the Haar measure.

The main assertion of Stochastic Quantization is that the path integral correlation functions of the field theory, computed with the Haar measure, can be traded for stochastic time averages in the asymptotic $\tau \to \infty$ limit:

$$\frac{1}{Z}\int \mathcal{D}U_k\, \mathcal{O}[U_k(\mathbf{x})] \exp(-S) = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \mathrm{d}\tau'\, \mathcal{O}[U_k(\mathbf{x},\tau')]\ , \tag{3.2}$$

where $\mathcal{O}$ is some observable. NSPT is obtained by expressing the solution of the Langevin equation as a power series in the coupling constant and by numerically integrating the hierarchy of equations that results from inserting this expansion into eq. (3.1). In our notation the expansion reads [30, 31]

$$U_k(\mathbf{x}) = 1 + \sum_{i=1}^{N} \beta^{-\frac{i}{2}}\, U_k^{(i)}(\mathbf{x})\ , \tag{3.3}$$

in which N is the highest order one wants to reach in the computation. In our case we need to expand the field up to β^{-4}, that is $N = 8$. Note that since eq. (3.1) guarantees unitarity by construction, the terms $U_k^{(i)}(\mathbf{x})$ will automatically inherit the corresponding properties.

While the spirit of Parisi's and Wu's original paper [32] was to offer a possibility for performing perturbative computations without gauge fixing, gauge fixing can naturally be added to the framework through the Faddeev-Popov mechanism, like to lattice gauge theory in general. The partition function is written as

$$Z = \int \mathcal{D}U_k \det \Delta_{\mathrm{FP}} \exp\left(-S_{\mathrm{L}} - S_{\mathrm{GF}}\right)\ , \tag{3.4}$$

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where $S_{\rm L}$ is the Wilson action in eq. (2.5). For $S_{\rm GF}$, we choose the standard covariant form,

$$S_{\rm GF} = \frac{\beta}{4N_{\rm c}\alpha} \sum_{\mathbf{x},A} \left[\hat{\partial}_k^L \phi_k^A(\mathbf{x})\right]^2 , \tag{3.5}$$

where α is the gauge parameter; $\hat{\partial}_k^L$ is the left difference operator; we have defined Lie algebra valued fields ϕ_k through $U_k = \exp(i\phi_k)$; and we have written $\phi_k = \phi_k^A T^A$. The naive continuum limit is obtained through the identification $\phi_k = ag_3 A_k$. Once we enforce the expansion in eq. (3.3) for the links, the Lie algebra valued fields are expanded as well: one simply needs the Taylor series for $\phi_k = -i\ln[1+\delta U_k]$.

Note that in standard lattice perturbation theory eq. (3.4) is further modified by writing the functional integration in terms of ϕ_k. On the contrary, we directly expand in terms of the link variables U_k (cf. eq. (3.3)), in terms of which eq. (3.1) is formulated. Therefore we do not need to add any "measure term" to the action. Nevertheless, the values of various operator expectation values remain identical, order by order in β^{-1}, to those in the standard perturbative framework.

The Faddeev-Popov operator corresponding to the gauge function in eq. (3.5) reads $\Delta_{\rm FP} \equiv -\hat{\partial}_k^L \hat{D}_k[\phi]$. Since $S_{\rm GF}$ is expressed in terms of Lie algebraic fields, $\hat{D}_k[\phi]$ comes from evaluating the response of the field ϕ_k to a gauge transformation. The latter is defined in terms of the link variables U_k, so it does not come as a surprise that $\hat{D}_k[\phi]$ is not expressed in closed form. It can however be written as a perturbative expansion, whose first terms read (see e.g. Ref. [37], whose notations we follow)

$$\hat{D}_k[\phi](\mathbf{x}) = \left[1 + \frac{i}{2}\Phi_k(\mathbf{x}) - \frac{1}{12}\Phi_k^2(\mathbf{x}) - \frac{1}{720}\Phi_k^4(\mathbf{x}) - \frac{1}{30240}\Phi_k^6(\mathbf{x}) + \mathcal{O}\left(\Phi_k^8\right)\right] \hat{\partial}_k^R + i\Phi_k(\mathbf{x}) , \tag{3.6}$$

where $\hat{\partial}_k^R$ is the right difference operator, while Φ_k is the field in the adjoint representation,

$$\Phi_k(\mathbf{x}) \equiv \phi_k^A(\mathbf{x})\, F^A , \qquad (F^A)_{BC} = -if^{ABC} . \tag{3.7}$$

In order to set up the proper Langevin equation, we finally rephrase the Faddeev-Popov determinant as a new contribution to the action, $\det \Delta_{\rm FP} = \exp(\mathrm{Tr}\, \ln \Delta_{\rm FP})$.

3.2 Mass regulator

We still need to add a mass regulator to the gauge-fixed framework. We note that the same mass is given to the gluon and the ghost fields in the continuum computation we want to match to. We therefore modify the total action to become

$$\tilde{S} \equiv \tilde{S}_{\rm L} + S_{\rm GF} + \tilde{S}_{\rm FP} . \tag{3.8}$$

The gluonic action has been modified by a mass term,

$$\tilde{S}_{\rm L} \equiv S_{\rm L} + \frac{\beta\,(am)^2}{4N_{\rm c}} \sum_{\mathbf{x}} \phi_k^A(\mathbf{x})\phi_k^A(\mathbf{x}) , \tag{3.9}$$

and the Faddeev-Popov action reads

$$\tilde{S}_{\rm FP} \equiv -\mathrm{Tr}\, \ln \tilde{\Delta}_{\rm FP} , \quad \tilde{\Delta}_{\rm FP} \equiv -\hat{\partial}_k^L \hat{D}_k[\phi] + (am)^2 . \tag{3.10}$$

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Even if we never make use of ghost fields in our approach, eq. (3.10) amounts to giving them a mass, as is clear from writing the operator $\tilde{\Delta}_{\rm FP}$ in Fourier-representation.

3.3 Some technical implementation issues

To treat the Faddeev-Popov determinant as a part of the action means that, because of the Langevin equation, one has to face the quantity $\nabla_{k,\mathbf{x}}\tilde{S}_{\rm FP} = -\nabla_{k,\mathbf{x}}{\rm Tr}\,[\ln\tilde{\Delta}_{\rm FP}] = -T^A{\rm Tr}\,[\nabla^A_{k,\mathbf{x}}\tilde{\Delta}_{\rm FP}\ \tilde{\Delta}^{-1}_{\rm FP}]$, with $\tilde{\Delta}_{\rm FP}$ from eq. (3.10). We follow the procedure proposed in Ref. [38] (see also Ref. [34]). Our study is actually the first practical implementation, but the essential ingredients are the same as for the treatment of the fermionic determinant in unquenched NSPT, as discussed in Ref. [31].

By introducing an extra gaussian noise ξ normalized as $\langle\xi^*_k\xi_n\rangle_\xi = \delta_{kn}$, one can substitute[1]

$$-\,{\rm Tr}\,[\nabla^A_{k,\mathbf{x}}\tilde{\Delta}_{\rm FP}\ \tilde{\Delta}^{-1}_{\rm FP}] \longrightarrow -\langle{\rm Re}\{\xi^*_k(\nabla^A_{k,\mathbf{x}}\tilde{\Delta}_{\rm FP})_{kl}(\tilde{\Delta}^{-1}_{\rm FP})_{ln}\xi_n\}\rangle_\xi\ . \tag{3.11}$$

Taking the real part would be unnecessary after the average $\langle\ldots\rangle_\xi$, but we find it convenient to impose it already before the averaging. The advantage of the form introduced is made clear by rewriting

$$\xi^*_k(\nabla^A_{k,\mathbf{x}}\tilde{\Delta}_{\rm FP})_{kl}(\tilde{\Delta}^{-1}_{\rm FP})_{ln}\xi_n \equiv \xi^*_k(\nabla^A_{k,\mathbf{x}}\tilde{\Delta}_{\rm FP})_{kl}\psi_l\ , \tag{3.12}$$

where $\psi \equiv \tilde{\Delta}^{-1}_{\rm FP}\,\xi$. In NSPT, then, we need to compute

$$\psi^{(i)} \equiv (\tilde{\Delta}^{-1}_{\rm FP})^{(i)}\ \xi\ . \tag{3.13}$$

It is worth stressing that the noise ξ has no power expansion, while the field ψ (like any other field in NSPT) is expanded because of the power expansion of $\tilde{\Delta}^{-1}_{\rm FP}$ (which is a function of the fields ϕ, i.e. of the fields U).

That eq. (3.13) can be evaluated efficiently within the NSPT framework stems from the expansion of the operator $\tilde{\Delta}^{-1}_{\rm FP}$. Once a generic matrix M is given as a power expansion,

$$M = M^{(0)} + \sum_{i=1}^{\infty}\beta^{-\frac{i}{2}}M^{(i)}\ , \tag{3.14}$$

the expansion for its inverse reads

$$M^{-1} = [M^{(0)}]^{-1} + \sum_{i=1}^{\infty}\beta^{-\frac{i}{2}}[M^{-1}]^{(i)}\ . \tag{3.15}$$

Here the non-trivial terms are obtained through a simple recursive relation:

$$\begin{aligned}
[M^{-1}]^{(1)} &= -[M^{(0)}]^{-1}M^{(1)}[M^{(0)}]^{-1}\ ,\\
[M^{-1}]^{(2)} &= -[M^{(0)}]^{-1}M^{(2)}[M^{(0)}]^{-1} - [M^{(0)}]^{-1}M^{(1)}[M^{-1}]^{(1)}\ ,\\
&\ldots\\
[M^{-1}]^{(i)} &= -[M^{(0)}]^{-1}\sum_{j=0}^{i-1}M^{(i-j)}[M^{-1}]^{(j)}\ .
\end{aligned} \tag{3.16}$$

[1] Here k, l, n should be regarded as multi-indices, including space coordinates and colour.

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In our case this leads to

$$\begin{aligned}
\psi^{(0)} &= [\tilde{\Delta}^{(0)}_{\rm FP}]^{-1}\xi \,, \\
\psi^{(1)} &= -[\tilde{\Delta}^{(0)}_{\rm FP}]^{-1}\tilde{\Delta}^{(1)}_{\rm FP}\psi^{(0)} \,, \\
\psi^{(2)} &= -[\tilde{\Delta}^{(0)}_{\rm FP}]^{-1}\left[\tilde{\Delta}^{(2)}_{\rm FP}\psi^{(0)} + \tilde{\Delta}^{(1)}_{\rm FP}\psi^{(1)}\right] \,, \\
&\cdots \\
\psi^{(i)} &= -[\tilde{\Delta}^{(0)}_{\rm FP}]^{-1}\sum_{j=0}^{i-1}\tilde{\Delta}^{(i-j)}_{\rm FP}\psi^{(j)} \,.
\end{aligned} \tag{3.17}$$

Eq. (3.17) states that there is no actual matrix inversion to take. Indeed, $[\tilde{\Delta}^{(0)}_{\rm FP}]^{-1}$ is independent of the ϕ fields and its expression is well known: it is the (*would-be*) ghost propagator, diagonal in Fourier space. Note that the mass regulator makes it well-defined at every value of the momentum. As for the various orders $\psi^{(i)}$, they are naturally computed by iteration. At every order only one application of $[\tilde{\Delta}^{(0)}_{\rm FP}]^{-1}$ is needed: this propagator operates on a sum of already computed quantities (the lower order ψ's). While $[\tilde{\Delta}^{(0)}_{\rm FP}]^{-1}$ is diagonal in momentum space, all the other operators are almost diagonal in configuration space. This suggests the strategy of going back and forth from Fourier space via a Fast Fourier Transform. It remains to point out that also the expression for $\nabla^A_{k,\mathbf{x}}\tilde{\Delta}_{\rm FP}$ (and its power expansion) is substantially local, so that the big inner product in eq. (3.12) is not too difficult to deal with. Finally, $\tilde{S}_{\rm FP}$ and $\tilde{\Delta}_{\rm FP}$ are naturally written in the adjoint representation, so that one has to devise an efficient way of dealing with cascades of commutators of the ϕ fields.

In order to solve eq. (3.1) numerically, the stochastic time variable τ needs to be discretised: $\tau \equiv n a_\tau$, where n is an integer. We use different values of a_τ, average over each thermalised signal, and then extrapolate in order to get the value of the desired observable at $a_\tau = 0$. Eq. (3.1) is discretized in the standard way [34] which automatically preserves the unitarity of our degrees of freedom:

$$U_k(\mathbf{x},(n+1)a_\tau) = e^{-iF_k(\mathbf{x},na_\tau)[U,\eta]}\; U_k(\mathbf{x},na_\tau) \,, \tag{3.18}$$

where

$$F_k(\mathbf{x},na_\tau)[U,\eta] \;= a_\tau\nabla_{k,\mathbf{x}}\tilde{S}[U] + \sqrt{a_\tau}\,\eta_k(\mathbf{x},na_\tau) \,, \tag{3.19}$$

and we have assumed the normalization $\langle\eta(\mathbf{x},ma_\tau)\eta(\mathbf{y},na_\tau)\rangle_\eta = 2\delta_{\mathbf{xy}}\delta_{mn}$.

We note that eq. (3.18) is only accurate to first order in a_τ. As a consequence, if the action is written as a sum ($\tilde{S} \equiv \tilde{S}_{\rm L} + S_{\rm GF} + \tilde{S}_{\rm FP}$), one can to the same accuracy rewrite eq. (3.18) as

$$U_k(\mathbf{x},(n+1)a_\tau) = e^{-iF^{(2)}_k[U]}\;\left\{\; e^{-iF^{(1)}_k[U,\eta]}\; U_k(\mathbf{x},na_\tau)\;\right\} \,. \tag{3.20}$$

The advantage of this form is that from the point of view of program implementation, it is easier to first evolve the field by

$$F^{(1)}[U,\eta] = \; a_\tau\nabla_{k,\mathbf{x}}S_{\rm L} + \sqrt{a_\tau}\,\eta_k \tag{3.21}$$

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am	exact	NSPT
0.10	2.6579	2.6581(6)
0.15	2.6481	2.6476(6)
0.20	2.6379	2.6379(8)
0.40	2.5713	2.5710(8)
0.60	2.4714	2.4710(8)
0.80	2.3485	2.3481(7)
1.00	2.2119	2.2117(6)
1.20	2.0689	2.0681(5)
1.40	1.9251	1.9247(5)
1.60	1.7848	1.7846(4)

Table 1: Comparison between "exact" and NSPT results for the coefficient of β^{-1} for the observable in eq. (4.1). The lattice extent is $L = 8a$, except for $am = 0.10$, where it is $L = 10a$.

(i.e. the contribution coming from standard Wilson action plus gaussian noise) and then by

$$F^{(2)}[U] = a_\tau \nabla_{k,\mathbf{x}} (\tilde{S} - S_{\mathrm{L}}) \; . \tag{3.22}$$

Indeed, $F^{[2]}[U]$ depends on the U's only through the ϕ's, i.e. the first step is performed in terms of the U fields, the second in terms of the ϕ fields. It is also easy to realize that the first step can be implemented as a sequential sweep through the lattice, while the second one requires the construction of a *global* contribution (the inner product in eq. (3.12)).

4. Data analysis and results

Since this work is the first time that the Faddeev-Popov procedure was implemented in NSPT, much attention was devoted to reliability checks, which we describe in the first subsection. The second subsection is devoted to carrying out the limit in eq. (2.22). All the numerical values shown in the following were obtained with $N_{\mathrm{c}} = 3$.

4.1 Consistency checks

The first checks were performed against the theory without gauge fixing, both in 3 and in 4 dimensions. In other words, we set $m \equiv 0$, and checked that we reproduce gauge invariant results for the plaquette expectation value, irrespective of the gauge parameter α used. In particular, we made sure that, for fixed volumes ($V \equiv L^3$), we could reproduce the 3d results in Ref. [26], up to 4-loop order.

We then plugged the mass terms in. A first test was that the leading $\mathcal{O}(\beta^{-1})$ contribution to the plaquette expectation value appearing in eq. (2.22),

$$\langle 1 - \tilde{\Pi}_{12} \rangle_{\mathrm{L}} = \frac{1}{\beta} \frac{d_A}{3} \left[1 - \frac{(am)^2}{N^3} \sum_{n_i=1}^{N} \frac{1}{\sum_{i=1}^{3} 4 \sin^2(\pi n_i / N) + (am)^2} \right] + \mathcal{O}\Big(\frac{1}{\beta^2}\Big) \; , \tag{4.1}$$

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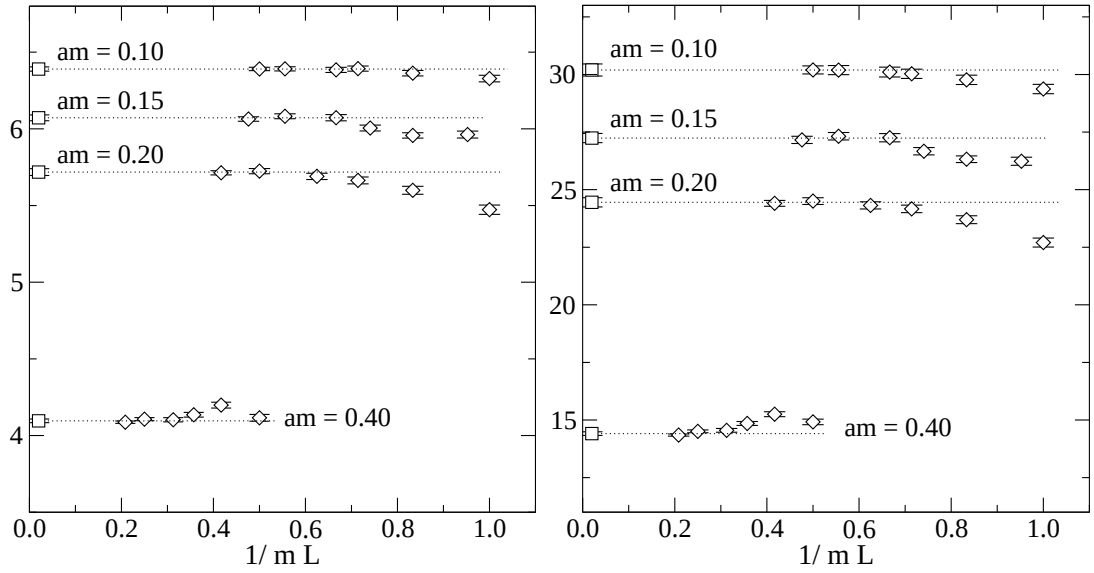

Figure 1: Examples of finite-volume values (open diamonds) and infinite-volume extrapolations (open squares) for the coefficient of β^{-3} (left) and β^{-4} (right) in the mass regularized plaquette expectation value, eq. (4.1).

where $N \equiv L/a$, was correctly reproduced by the NSPT numerics. Table 1 shows the results for the coefficient of β^{-1} and gives an idea of the size of errors. It also lists all the mass values that we will use in the following.

As a final preparation we checked, as already explained in Ref. [39], that if the volume is kept finite, and a fit in $(am)^2$ is performed for measurements carried out with the masses shown in Table 1, then the intercept with the axis $am = 0$ agrees with previous results in the massless theory, obtained without gauge fixing [26].

4.2 Detailed analysis

After the consistency checks, we turn to the actual analysis of eq. (2.22). It consists of two steps. First, for any given value of m, we need to carry out an infinite-volume extrapolation. Second, the infinite-volume extrapolations need in turn to be extrapolated to $m \to 0$, as dictated by eq. (2.22). Both of these extrapolations are rather delicate so let us describe the procedure that we adopt in some detail.

For the infinite-volume extrapolations, we expect that the dependence on L is exponentially small, once the volume is large enough. Concretely, inspecting the 1-loop expression in eq. (4.1) for $am \ll 1$ but at finite volumes, suggests that for $mL \gg 1$ the behaviour is

$$\langle 1 - \tilde{\Pi}_{12} \rangle_{\rm L} \sim \gamma_0 + \frac{1}{mL} \Big[\gamma_1 \exp(-mL) + \gamma_2 \exp(-\sqrt{2} mL) + \cdots \Big] \,, \tag{4.2}$$

where $\gamma_1, \gamma_2, \ldots$ have all the same sign. However, for our larger masses $am \gtrsim 0.40$ (cf. Table 1), the volume dependence appears in fact to be dominated by discretization effects not contained in eq. (4.2). Moreover, at higher loop orders other structures also appear and it is not clear *a priori* how large mL has to be for them to remain negligible (note

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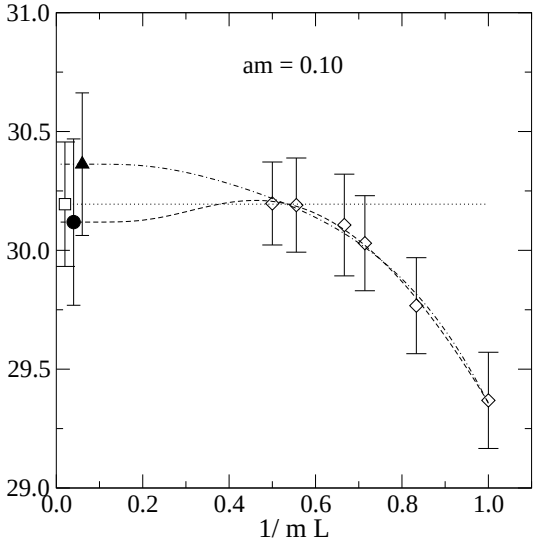

Figure 2: An example of different infinite-volume extrapolations for the coefficient of β^{-4} in the mass regularized plaquette expectation value, for $am = 0.10$. The open square denotes the extrapolation described in the text; the closed circle is an extrapolation according to eq. (4.2) (with unconstrained coefficients); and the closed triangle assumes yet another phenomenological fit form, $\delta_0 + \exp(-mL)[\delta_1 + \delta_2(mL)^{-4}]$.

that for the important small masses $am \lesssim 0.20$ we are only able to go up to $mL \sim 2$, cf. figure 1).

For these reasons, we adopt a practical procedure in the following whereby we increase the volume until no volume dependence is seen within the error bars, and then fit a constant to data in this range. To be conservative, the resulting error bars are multiplied by a factor two. The original "raw" data at finite volumes, and the corresponding infinite-volume extrapolations obtained with the recipe just described, are illustrated in figure 1 for a few masses. However, we have also tried other procedures, like a fit according to eq. (4.2) or to different phenomenological forms ruled by decaying exponentials multiplied by polynomial prefactors. Figure 2 gives an idea of the effects of these variations on the infinite-volume extrapolations. It can be seen that our doubled error bars can cover all the variations.

Given the infinite-volume extrapolations, we can carry out the extrapolation $am \to 0$. Motivated again by a 1-loop analytic computation (eq. (4.1) for $N \to \infty$), we use an ansatz allowing for any positive powers of am. There is the problem, however, that the data points are more precise at larger masses: the absolute errors decrease roughly as $\sim 1/(am)^2$, i.e. vary by two orders of magnitude. Thus large masses tend to dominate the fit, while the most important region should be that of small masses.

We confront this situation in the following way. First of all, we allow for a high-order fit polynomial, and monitor the stability of the results, and the χ^2-value of the fit, with respect to the order of polynomial, as well as the number of masses that are taken into account. Second, we consider both the regular χ^2-function, and modified ones where the errors are weighted by am or by $(am)^2$, so as to make the contributions of the different

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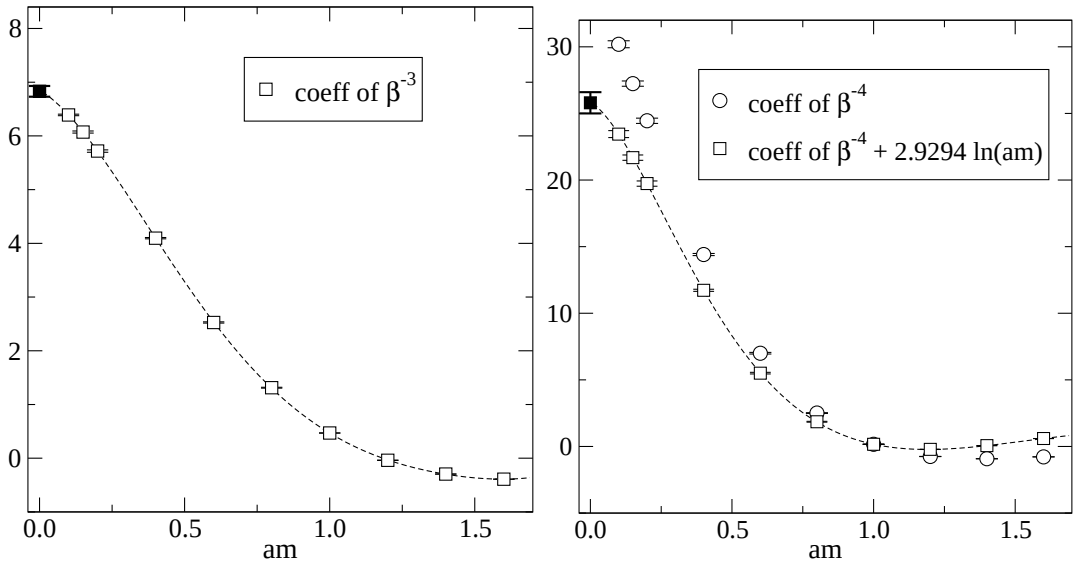

Figure 3: The $am \to 0$ extrapolations (dashed lines, with the intercepts with the axis $am = 0$ shown with closed squares) through the infinite-volume extrapolated data points (open squares), for the coefficient of β^{-3} (left) and β^{-4} (right) in the mass regularized plaquette expectation value.

data points more balanced. All of these fits generally show an extremum as the order of the fit polynomial is increased, with the extremal values differing by less than the statistical errors. Examples of the fits are shown in figure 3.

In order to benchmark this strategy, let us first apply it at 1-loop, 2-loop, and 3-loop levels. We obtain

$$\lim_{m\to 0}\left\{\lim_{L\to\infty}\langle 1-\tilde{\Pi}_{12}\rangle_{\rm L}\Big|_{\beta^{-1}}\right\} = 2.672 \pm 0.008\ , \tag{4.3}$$

$$\lim_{m\to 0}\left\{\lim_{L\to\infty}\langle 1-\tilde{\Pi}_{12}\rangle_{\rm L}\Big|_{\beta^{-2}}\right\} = 1.955 \pm 0.016\ , \tag{4.4}$$

$$\lim_{m\to 0}\left\{\lim_{L\to\infty}\langle 1-\tilde{\Pi}_{12}\rangle_{\rm L}\Big|_{\beta^{-3}}\right\} = 6.83 \pm 0.10\ . \tag{4.5}$$

These numbers are to be compared with the known results in eqs. (2.12)–(2.14); we find perfect agreement within error bars.

We then repeat the same procedure at 4-loop level. As shown by eq. (2.22), the extrapolation $am \to 0$ can only be carried out after the subtraction of the logarithmic IR term. The fit shown in figure 3 produces

$$\tilde{B}_{\rm L}(1) = \left(\frac{2\pi^2}{27}\right)^2 \times (25.8 \pm 0.8) = 13.8 \pm 0.4\ . \tag{4.6}$$

Inserting into eq. (2.20), a significant cancellation takes place, and we obtain

$$B_{\rm G} = -0.2 \pm 0.4^{\rm (MC)} \pm 0.4^{\rm (NSPT)}\ , \tag{4.7}$$

where the error "MC" originates from the lattice Monte Carlo simulations in Ref. [17], and the error "NSPT" from the analysis in the present paper. Eq. (4.7) is our final result. Let

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us also record the $N_{\rm c} = 3$ values for the constant C_4' in eq. (2.8) and c_4' in eq. (2.10),

$$C_4' = 10.9 \pm 0.3^{(\rm NSPT)} \;, \quad c_4' = 7.0 \pm 0.3^{(\rm NSPT)} \;. \tag{4.8}$$

5. Conclusions and perspectives

We have demonstrated in this paper the feasibility of determining the non-perturbative constant $B_{\rm G}$, defined through eq. (2.3), by combining previous lattice Monte Carlo results [17] with a 4-loop perturbative matching step. The matching involves a comparison of a continuum $\overline{\rm MS}$ computation [23] with the corresponding lattice regularized computation. The latter we have carried out with the help of Numerical Stochastic Perturbation Theory (NSPT). The final estimate for the new matching coefficient (with two different conventions) is shown in eq. (4.8). Taking into account the Monte Carlo results, the final estimate for $B_{\rm G}$ is shown in eq. (4.7).

We note that within the current errors, $B_{\rm G}$ is consistent with zero. This is a matter of conventions, however; for instance, had we not made the arbitrary choice of including the factor 2 inside the logarithm in eq. (2.3), the corresponding constant would be non-zero by a significant amount.

Given that the physical pressure of hot QCD is numerically fairly sensitive to $B_{\rm G}$ [12], it would of course be desirable to improve on the accuracy of $B_{\rm G}$, both on the lattice Monte Carlo and on the NSPT sides. For instance, it would be interesting to repeat the current study with traditional techniques [27]. Moreover, it should in principle be possible to carry out the matching leading to eqs. (4.8) by using a finite volume rather than a mass as an infrared regulator; for this approach the NSPT side exists already [26], but the 4-loop $\overline{\rm MS}$ computations of Ref. [23] would have to be repeated with techniques discussed at 1-loop level for instance in Refs. [40].

Apart from these challenges, there is now an ever more compelling case for determining the last remaining purely perturbative $\mathcal{O}(g^6)$ term, i.e. the constant denoted by $\beta_{\rm E1}$ in Ref. [12]. Only after this has been added does the $\bar{\mu}$-dependence of eq. (2.3) get cancelled, such that the full physical pressure is scale-independent, as it has to be.

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